Instability of weakly nonlinear chaotic structures

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A linear oscillator driven by periodic perturbation is considered. An infinite connected chaotic structure in the phase plane emerges when the perturbation is of the form of a periodic δ function and the exact resonance condition is fulfilled [Zaslavsky, Sagdeev, D. A. Usikov, and A. A. Chernikov, *Weak Chaos and Quasi-Regular Patterns* (Cambridge University Press, Cambridge, 1991)]. These structures are shown to be unstable and completely destroyed if the duration of the perturbation is arbitrarily short but finite.

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A linear oscillator driven by a periodic nonlinear perturbation is often used as a dynamical model for some physical problems. Its motion is described by the Hamiltonian

$$H(x,p,t) = \frac{p^2+\omega_0^2 x^2}{2} + \varepsilon\, V(x,p,t)\,, \qquad (1)$$

where the perturbation $V(x,p,t+T) = V(x,p,t)$ is a time-periodic function. In Eq. (1) the nonlinearity depends (as $\varepsilon \to 0$) on the weak perturbation only and, if a resonance occurs, we have a weakly nonlinear resonance system on which the extension of the Kolmogorov-Arnol'd-Moser theory is impossible. The dynamics of the systems, which may be very unexpected, was extensively studied (see, e.g., [1]).

Let the perturbation in Eq. (1) contain only one of the harmonics. Then

$$H(x,p,t) = \frac{p^2+\omega_0^2 x^2}{2} + \varepsilon\, \cos(x - \Omega\, t)\,. \qquad (2)$$

If we put $\omega_0 = 0$ the model describes a single strongly nonlinear resonance (SNR) which is completely integrable with no trace of a chaos [2]. Yet, for any $\omega_0 \neq 0$ (as $\varepsilon \to 0$), nonlinearity becomes weak and the motion drastically changes. If Ω/ω_0 is an integer, the model describes a single weakly nonlinear resonance (WNR) which has a very complicated chaotic component [3].

In the present paper we are going to discuss in detail another case of Eq. (1) with an infinite number of harmonics

$$H(x,p,t) = \frac{p^2+\omega_0^2 x^2}{2} + \varepsilon\, \cos x\, \delta_T(t)\,, \qquad (3)$$

where $\delta_T(t)$ is the periodic δ function with period T and the perturbation parameter $\varepsilon << 1$ is small. The model (3) may represent the motion of a charged particle in both a magnetic field (Larmor's frequency ω_0) and the field of a perpendicularly propagating wave packet [1]. If we put $\omega_0 = 0$ the model describes a strongly nonlinear system with an infinite set of interacting resonances and their chaotic layers. For a sufficiently small perturbation, $\varepsilon << 1$, the layers of different resonances are separated from each other by stable invariant tori and an unbounded motion is impossible [4]. But for any $\omega_0 \neq 0$ and $T = 2\pi/n\omega_0$ with any integer n an infinite and uniform connected chaotic web emerges on the phase plane [1]. The unbounded motion of a particle along this web is possible. Note that the web also exists when the coefficient of the δ function in Eq. (3) is not purely $\cos x$, but is given an explicit sinusoidal time dependence [5] or any periodic function $f(x)$ [6]. The web of Eq. (3) is unstable against detuning from the exact resonance condition ($2\pi/T - n\omega_0 \neq 0$), as was shown in [7] (see also [1]). We study here another kind of instability due to a finite kick's width.

Let us replace the δ function in Eq. (3) by another one $F(t)$ which has the form of a periodic rectangular function of length Δ and height $1/\Delta$, located in the middle of every period T. Then

$$\begin{aligned} H_n(x,p,t) &= \frac{p^2+\omega_0^2 x^2}{2} + \varepsilon\, \cos x\, F(t) \\ &= \omega_0\, I + \varepsilon\, \cos(\rho\, \cos\, \theta)\, F(t)\,, \end{aligned} \qquad (4)$$

where

$$F(t) = \frac{a_0}{2} + \sum_{k\geq 1} a_k\, \cos(kn\omega_0 t)\,,$$

$$a_k = \frac{n\omega_0}{\pi}(-1)^k \frac{\sin(kn\omega_0\Delta/2)}{(kn\omega_0\Delta/2)}\,; \qquad k = 0,\, 1,\, 2, \ldots\,.$$

In Eq. (4) $x = \rho\cos\theta$, $p = -\rho\,\omega_0 \sin\theta$, and $\rho = (2I/\omega_0)^{1/2}$ is the amplitude of the unperturbed oscillations. The exact resonance condition $T = 2\pi/n\omega_0$ with some integer n is supposed to be fulfilled. This condition is the only one and the model describes a single WNR [compare with Eq. (2)].

Introducing a new slow phase $\varphi = \theta - \omega_0\, t$, a new time $\tau = \varepsilon\, n\, \omega_0\, t/2\,\pi$, expanding the perturbation in a series of Bessel functions, and averaging over the fast oscillations, we arrive at a first-order resonance Hamiltonian (for more detail see [1, 2]):

$$\begin{aligned} \widetilde{H}_{\Delta,n}(I,\varphi,\Delta) = J_0(\rho) & \\ +2 \sum_{k\geq 1} & (-1)^{k+\frac{kn}{2}} J_{kn}(\rho) \\ & \times \frac{\sin(kn\omega_0\Delta/2)}{(kn\omega_0\Delta/2)} \cos(k\, n\, \varphi)\,, \end{aligned} \qquad (5)$$

where kn are even. For $\Delta = 0$ the perturbation has the form of a periodic δ function as in Eq. (3) and we will use for this case a special symbol $\widetilde{H}_{\delta,n}$:

$$\widetilde{H}_{\delta,n}(I,\varphi) = \widetilde{H}_{\Delta,n}(I,\varphi,\Delta=0) = J_0(\rho) + 2\sum_{k\geq 1}(-1)^{k+\frac{kn}{2}} J_{kn}(\rho)\cos(k\,n\,\varphi)\,, \tag{6}$$

with kn even.

Expanding $\sin(kn\omega_0\Delta/2)/(kn\omega_0\Delta/2)$ and regrouping terms in Eq. (5) we obtain

$$\widetilde{H}_{\Delta,n}(I,\varphi,\Delta) = \widetilde{H}_{\delta,n}(I,\varphi) + \sum_{m\geq 1}\frac{A_{2m}(I,\varphi)}{(2m+1)!}\left(\frac{\omega_0\Delta}{2}\right)^{2m}, \tag{7}$$

$$A_{2m}(I,\varphi) = 2\sum_{k\geq 1}(-1)^{\frac{kn}{2}+k+m} J_{kn}(\rho)(kn)^{2m}\cos(k\,n\,\varphi)\,,$$

where kn are even. Using $2m$ multiple integration of $A_{2m}(I,\varphi)$ with respect to φ and representing the results in terms of $\widetilde{H}_{\delta,n}(I,\varphi)$ and its derivatives, we obtain an interesting relation between the two Hamiltonians:

$$\widetilde{H}_{\Delta,n}(I,\varphi,\Delta) = \sum_{m\geq 0}\frac{(\omega_0\Delta/2)^{2m}}{(2m+1)!}\frac{\partial^{2m}}{\partial\varphi^{2m}}\widetilde{H}_{\delta,n}(I,\varphi)\,. \tag{8}$$

A part of the infinite and uniform web for the resonant case $n = 4$ and $F(t) = \delta_T(t)$ is reproduced in Fig. 1. This so-called "kicked Harper model" (KHM) is extensively studied now by many researchers [8].

A compact form of the first-order resonance Hamiltonian (6) may be derived by means of the following identity ([9], $m \geq 1$ is any integer):

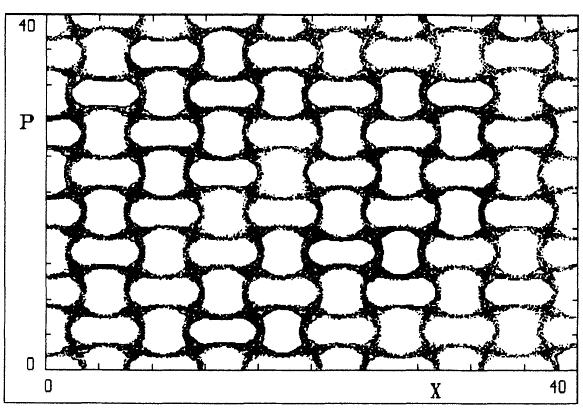

FIG. 1. Computer simulation of model Eq. (4) with $n = 4$, $\omega_0 = 1$, and $\varepsilon = \pi/2$. The perturbation $F(t)$ is a periodic δ function. X, P are slow variables (9) shown at $t = 0 \bmod T$ with initial values in the vicinity of the saddle point $X = P = \pi/\sqrt{2}$.

$$J_0(\rho) + 2\sum_{k\geq 1}(-1)^{mk} J_{2mk}(\rho)\cos(2mk\vartheta) = \frac{1}{m}\sum_{j=0}^{m-1}\cos\left[\rho\cos\left(\frac{j\pi}{m}+\vartheta\right)\right].$$

For KHM, by setting in this identity $m = 2$, $\vartheta = \varphi + \pi/4$ and using slow variables

$$X = \rho\cos\varphi\,,\quad P = -\rho\omega_0\sin\varphi\,, \tag{9}$$

we obtain

$$\widetilde{H}_{\delta,4}(X,P) = \cos\left(\frac{X}{\sqrt{2}}\right)\cos\left(\frac{P}{\sqrt{2}}\right). \tag{10}$$

It coincides with the results of [1] up to a rotation of the Cartesian coordinate axes. Exactly the same form has the first-order resonance Hamiltonian for a model of Eq. (2) with the only harmonics sufficiently far from the origin $X = P = 0$. Its structure and instability were considered in detail in [3].

This resonance structure is characterized by an infinite lattice of periodic trajectories (fixed points on a plane X, P) both stable $[\sin(X/\sqrt{2}) \approx \sin(P/\sqrt{2}) \approx 0]$ and unstable $[\cos(X/\sqrt{2}) \approx \cos(P/\sqrt{2}) \approx 0]$, the latter being connected by separatrices with their chaotic layers. The infinite WNR structure in Fig. 1 is qualitatively different from the SNR one. As is well known, the latter consists of a "chain of islands" which extends in the direction of phase variable only and is strictly bounded.

The SNR structure is universal and stable under sufficiently weak perturbations [4]. Unlike this, the WNR structure is neither universal (the structure is different for different n) nor stable. The first example for this was a model Eq. (2) [3].

To see the instability of the KHM structure let us add to Eq. (10) a term linear in X:

$$\widetilde{H}_{\delta,4}(X,P) = \cos\left(\frac{X}{\sqrt{2}}\right)\cos\left(\frac{P}{\sqrt{2}}\right) + aX\,. \tag{11}$$

Then the vertical separatrices (X =const) all remain unchanged but the horizontal ones (P =const) are destroyed because of the difference $\Delta\widetilde{H}_{\delta,4}$ in $\widetilde{H}_{\delta,4}$ between the two neighboring fixed points. Remarkably, an arbitrarily small perturbation ($a \to 0$) qualitatively changes the structure by making all the rows of resonant cells disconnected by narrow vertical gaps. For small $a > 0$ the width of a gap at $X = 3\pi/\sqrt{2} \bmod 2\sqrt{2}\pi$ and $P = 0 \bmod 2\sqrt{2}\pi$ is

$$\Delta X \approx 2\left|\frac{\Delta\widetilde{H}_{\delta,4}}{\partial\widetilde{H}_{\delta,4}/\partial X}\right| \approx 2\pi\,|a|\,. \tag{12}$$

The motion inside a gap is unbounded in P. One can realize such an "accelerating regime" for KHM by adding to the Hamiltonian (4) the time-dependent resonant perturbation

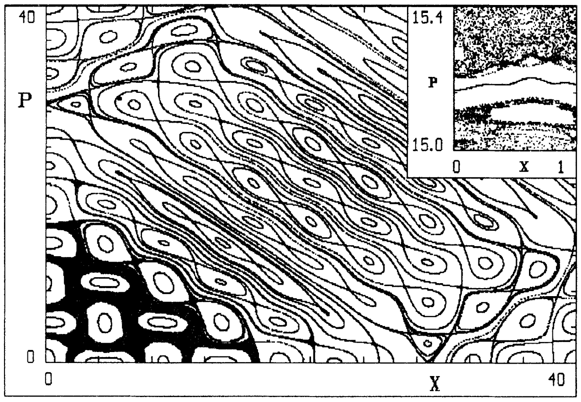

FIG. 2. The same as in Fig. 1 with a finite $\Delta/T = 0.2$. Insert: enlarged part of the first open gap and an invariant trajectory inside it.

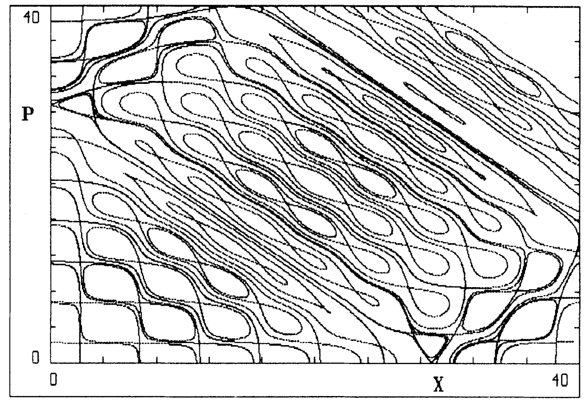

FIG. 3. The levels of the resonance Hamiltonian $\widetilde{H}_{\Delta,4}$ using the first 15 terms in Eq. (5). All parameters are the same as in Fig. 2.

$$\Delta H_4 = \frac{\pi a}{\omega_0} x \cos(\Omega t)\,, \tag{13}$$

where $\Omega = \omega_0$ (the case of a linear resonance in nonlinear webs, see [3]).

Now, let the kick in Eq. (4) be of finite width Δ. The phase portrait of the KHM for $\Delta/T = 0.2$ is given in Fig. 2 (the other parameters are as in Fig. 1). One may see the qualitative change in the phase plane structure: the infinite lattice of fixed points remains almost unchanged but the chaotic web is completely destroyed by the extremely complicated net of gaps. In Fig. 3 the levels of the resonance Hamiltonian $\widetilde{H}_{\Delta,4}$ (5) are shown. Remarkably, the analytical construction reproduces well the most principal features of the KHM phase structure of Fig. 2 (without chaotic components).

To clarify the origin of the gaps in Figs. 2 and 3, we use the relation Eq. (8) between the two resonance Hamiltonians $\widetilde{H}_{\Delta,4}$, $\widetilde{H}_{\delta,4}$ and Eq. (10) for the latter. The following calculations will be simplified if we restrict ourselves to the saddle points where

$$\cos(X/\sqrt{2}) \approx \cos(P/\sqrt{2}) \approx 0;$$

$$\sin(X/\sqrt{2}) \approx \sin(P/\sqrt{2}) \approx \pm 1\,.$$

In the saddle points of system (4) with $n = 4$ we have

$$\widetilde{H}_{\Delta,4}(X,P) \approx -\frac{XP}{6}\left(\frac{\omega_0\Delta}{2}\right)^2 + \frac{XP}{120}(X^2 + P^2 + 7)\left(\frac{\omega_0\Delta}{2}\right)^4 - \cdots\,. \tag{14}$$

This expression shows that the origin of the gaps in Figs. 2 and 3 is related to the difference in $\widetilde{H}_{\Delta,4}$ between the two neighboring fixed points. We can estimate the width of the gap from Eq. (14) and more accurately from Eq. (5). For several low lying (nearest to the origin $X = P = 0$) gaps in Fig. 3 both approximations give close results.

The dependence of the resonance Hamiltonian $\widetilde{H}_{\Delta,4}$ on X, P leads also to a change of the frequencies of small oscillations and to a decrease in the width of chaotic layers when moving away from the origin. For low gaps the chaotic layers may close the gaps completely but from some distance the gaps remain open and block the diffusion (the first open gap is shown in Fig. 2).

In conclusion, we note that the weakly nonlinear chaotic structure is an infinite connected web only if the perturbation has the form of the δ function and the exact resonance condition is fulfilled. As was shown above, the web is not stable. Under a weak additional perturbation or for a finite kick's width it becomes disconnected by many narrow gaps.

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